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Some notes on the singular manifold method: several manifolds and constraints

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Abstract. Some questions arising when various modifications are made to the singular manifold method are considered. The solutions are shown to lie outside the framework of the singular analysis. The approach is illustrated by a number of examples with ordinary differential equations. Further perspectives and the question of optimization of the calculations are also discussed.

1. Introduction

Although the singular manifold method or WTC approach [1] was introduced as a natural generalization of the ordinary Painlevé test to partial differential equations (PDEs) in the context of the necessary condition for their integrability [2] (more precisely, the connection between nonlinear PDEs associated with the linear Gel'fand–Levitan–Marchenko integral equation and the Painlevé property of their scaling-type self-similar solutions), nevertheless already in [1] the overwhelming majority of the results concern the use of the related functional series for deriving various relations and structures, such as the Bäcklund transformations and Lax pairs, for the equations under consideration rather than the Painlevé property itself. In the succeeding two decades several works refining the singular analysis have appeared (see, e.g., [3–6]), however, as before the singular manifold approach owes its popularity mainly to applications of those truncated expansions and the simplicity of their construction.

Part of the original theory does not demand any assumptions on the type of singularities and can be successfully applied for 'non-integrable' partial differential equations [7] and ordinary differential equations (ODEs). Moreover, to expand its practicality both for integrable and non-integrable cases a number of modifications have been made. Among them are the use of Weierstrass or elliptic functions [8], the Painlevé–Darboux transformations [9], 'double singular manifold methods' [10–12], and various ways for introducing judiciously chosen terms [13, 14]. Some of those modifications are compatible with the Painlevé analysis as shown in [15] and others simply break down its basic postulates, but all turn out to be effective in reality.

The main goal of this paper is to show that all the above approaches are closely related and are of the same unified algebraic nature and to point how these ideas can be developed further.

First of all, we need to point out that the association of $f(x, t)$ with singularities in the truncated or infinite expansions

$$u(x, t) = \sum_{i=-p}^{m} w_i(x, t) \cdot f^i(x, t) \qquad p \in N \quad m = 0, +\infty$$

for solutions alone does not provide reason enough to equate coefficients at its powers to zero after substitution into a governing equation. For it to be correct to do so for such a series, it is necessary that the combinations of derivatives of f arising as factors at its powers should have a non-zero first term together with other coefficients zero up to some order in the related Taylor expansions. For infinite series this demand leads to ordinary Laurent series. On the other hand, the above expansions must be valid over the whole domain for our purposes. The answer becomes especially clear by taking into account the so-called invariant formalism [5]

$$u(x, t) = \sum_{i=p}^{m} w_i(x, t) \cdot V^i(x, t) \qquad p \in N \quad m = 0, -\infty \tag{1}$$

where in terms of the function f

$$V = \frac{f_x}{f} - \frac{1}{2}\left(\frac{f_{xx}}{f_x}\right)$$

such that

$$V_x = -V^2 - \tfrac{1}{2}S \tag{2}$$

$$V_t = CV^2 - C_x V + \tfrac{1}{2}(CS + S_{xx}) \tag{3}$$

$$S = S(x, t) \qquad C = C(x, t)$$

and

$$S_t + C_{xxx} + 2SC_x + CS_x = 0. \tag{4}$$

In such a manner the function V depends on an additional arbitrary parameter in comparison with C, S and w_i as a consequence. In other words, the form of such series and the fact that the singularities are movable turn out to be more essential here. All the aforesaid is important for our considerations.

This paper is organized as follows. In section 2 the approach allowing one to find both singular manifold equations and additional constraints to the corresponding singular function is proposed, and its realizations are discussed. The examples with one and two singular manifolds for some nonlinear ODEs from physical applications are presented in sections 3 and 4, respectively. Their computational aspects are illustrated in detail. In section 5 we discuss how the main ideas can be developed further, as well as the common features of the WTC approach and the so-called method of generalized separation of variables. In appendices A and B some bulk mathematical treatments with technical details are given for the above-mentioned examples.

2. Additional constraints in the singular manifold method

In a number of cases there can also exist an additional constraint to a singular manifold (the function V), or, in other words, the above free parameter in V appears to be bound. In this section we will show why such constraints may arise and how this problem can be solved rigorously.

First, let us presume that we have a differential equation and the generalized Laurent series (1) for its general solution, i.e. $m = -\infty$ and there are $k-1$ arbitrary functions in the expansion besides V, where k is the order of this equation. Also, assume that we consider some of its reductions. What form does the Laurent expansion take for it? Obviously, for the infinite series to pick out the solutions corresponding to the reduction, it is necessary to impose some constraints on the arbitrary functions. Two different situations should be distinguished, because such constraints may or may not affect the singular manifold function V (it is one more arbitrary function). In terms of truncated series and singular manifold equations to S and C, it means that aside from reductions corresponding directly to the reductions of the singular manifold equation, there are also others demanding a linkage between the functions V and S, C. In the first case the traditional WTC algorithm is fully suitable. Otherwise, in the general case, we are not entitled to equate the coefficients after substitution. Therefore, only expressions for particular solutions can be constructed in such a way. However, the relations associated with the truncated expansion of the original system can be restored if the form of this expansion is known. And it is easy to see that the principal parts of the Laurent expansions for the original equation and for reductions are coincident if the arbitrary functions (resonances) are placed above them. However, for this another technique should already be applied.

The next case is expansions with several singular functions. In the works [10, 11] Estévez and co-workers attempted to introduce such truncated expansions with two manifolds for integrable equations with different singular branches. In so doing, to apply the traditional method and cancel terms of the form $f_1^{-n_1} f_2^{-n_2}$ $(n_1, n_2 \in N)$, the validity of the linkage

$$\frac{1}{f_1 f_2} = \frac{a(x,t)}{f_1} + \frac{b(x,t)}{f_2}$$

was assumed. Soon it was demonstrated that for cases with two opposite branches the last problem could be solved by reducing to a one-manifold case via the new more convenient notation [12]. However, since a second manifold in such truncated expansions can present the rest of the one-manifold infinite series [15], it is necessary to be able to use the above expansion of any type and form perceiving that the case at hand is constrained manifolds.

Next, the approach for such cases will be presented in detail. In so doing, we restrict ourselves to cases when the expressions under consideration are polynomial at least with respect to V. Although the theory itself is valid in the general case, however, techniques for systems of univariate and multivariate polynomials will be essential for our computations. While the overwhelming majority of real nonlinear equations from applications lead exactly to such cases. Also, ordinary differential equations will be considered for simplicity. All the following are immediately generalized to cases with several independent variables, but a greater number of relations should be considered in so doing.

We will start with the first case of one constrained singular function V. Assume that there is some ODE

$$E(x, u, u_x, \ldots, u_{kx}) = 0 \qquad k \in N \tag{5}$$

of the above type, and some singular manifold equation

$$M(x, S, S_x, \ldots) = 0 \tag{6}$$

with the additional constraint to V

$$G(V; x, S, S_x, \ldots) = 0$$

associated with it and the truncated expansion

$$u = T(V; x, S, S_x, \ldots, S_{lx}) = 0 \qquad l \in N. \tag{7}$$

Then after substitution of this expansion into equation (5) it can be factored as

$$E = A(V; x, S, S_x, \ldots)\, G(V; x, S, S_x, \ldots) = 0. \tag{8}$$

Obviously, the only constraint can be imposed to one function V, and the condition of its invariance [16] or compatibility with the differential equations to V (2) and S (6) is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}G = \frac{\partial G}{\partial x} + V_x\frac{\partial G}{\partial V} + S_x\frac{\partial G}{\partial S} + \cdots = L(V; x, S, S_x, \ldots)\, G \tag{9}$$

where L has no singularities at $G = 0$. (Simply speaking, if $G = 0$ for some $x = x_0$ then $G = 0$ for any x.) However, in view of (9) all the total derivatives of E with respect to x are also factorizable as before, in particular,

$$\frac{\mathrm{d}}{\mathrm{d}x}E = \left(\frac{\mathrm{d}}{\mathrm{d}x}A + AL\right) G(V; x, S, S_x, \ldots) = 0. \tag{10}$$

Conversely, if one has

$$E = A\,G$$
$$\frac{\mathrm{d}}{\mathrm{d}x}E = B\,G$$

with some polynomials A, B and G in V, then because

$$\frac{\mathrm{d}}{\mathrm{d}x}E = \frac{\mathrm{d}A}{\mathrm{d}x}G + A\frac{\mathrm{d}G}{\mathrm{d}x}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}G = \frac{(B - \mathrm{d}A/\mathrm{d}x)}{A}G.$$

So, if A divides $B - \mathrm{d}A/\mathrm{d}x$ or does not divide G, then G satisfies the invariant condition (9).

Hence our problem is reduced to finding some common divisor for (8) and (10) as well as to determine the appropriate differential equation (6) for S needed for the existence of the former. There are a number of methods in the theory of polynomial systems for solving this problem. Among them are the Gröbner bases technique [17], Wu's zero-structure theorem [18], etc. In the frameworks of these approaches an initial system can be brought to a triangular form, such that V will be eliminated from one or several final equations, and the equations for S with its derivatives and the common divisor sought will be distinguished as a result. Alternatively, S and its derivatives can be simply treated as the coefficients of two univariate polynomials with respect to V, and the theorem on a resultant [19] can be applied to determine the condition for the existence of their common zeros. After that the related common divisor could already be easily calculated by means of one of the standard techniques. All these methods have their own advantages and disadvantages. For instance, in the simplest situations a resultant leads directly to the equation sought for S within one step. However, practically such situations are limited to ODEs and cases when no further evaluations are needed (see below). Thus, for PDEs already three equations $E = 0$, $\partial E/\partial x = 0$ and $\partial E/\partial t = 0$ should be investigated, and its generalization will be wasteful of computer time and memory and already gives rise to a set of equations for S. And in the latter case, when a resultant is not the final expression to be found, its use may not be the most effective and short way to achieve a result. In such cases the Gröbner bases technique appears to be considerably more effective. The approach itself is very powerful and flexible. It allows one to exercise control over every step to discard superfluous branches according to possible extra conditions or properties of variables. At the present time there are many modifications and optimization algorithms, and almost every modern computer algebra system contains an implementation of it. (A fine introduction to the method with many references can be found in [20, 21]. See also [22] about its application to a number of problems with multivariate polynomials and the implementation used in REDUCE.)

Before proceeding further, it is necessary to underline that in the general case the resulting condition for S (it is an ODE of order $k + l + 1$) constitutes some differential prolongation of the singular manifold equation sought. Or, in other words, the singular manifold equation of interest (6) may appear to be just the so-called intermediate integral [23] for it. Really, there are many equations with the same truncated expansion (7) and such that the ODE under consideration (5) is their reduction of the above-mentioned type. And all their singular manifold functions must satisfy that condition. While these singular manifold equations in the framework of the WTC approach are derived from some overdetermined sets of differential equations and may, in principle, in view of the forms of (5) and (7), be of any order up to $k + l + 1$ inclusive.

Now let us consider series with several manifolds. In this case the generalizations of relations (8)–(10) for a one-manifold case are as follows ($m \leqslant n$):

$$\frac{\mathrm{d}}{\mathrm{d}x} G_i = \sum_{j=1}^{m} L_{ij}(V_1, \ldots, V_n; x, S, S_x, \ldots)\, G_j \qquad i = \overline{1, m} \tag{11}$$

$$\frac{\mathrm{d}^i}{\mathrm{d}x^i} E = \sum_{j=1}^{m} A_{ij}(V_1, \ldots, V_n; x, S, S_x, \ldots)\, F_j = 0 \qquad i = \overline{0, +\infty}. \tag{12}$$

Also, it is not hard to investigate, respectively, the first $m + 1$ of them (12) via, for example, the Gröbner basis technique and construct some suitable generators $\{G_1, \ldots, G_m\}$ for (12) together with the related equation for S. Such generators alone, however, are not unique, and to choose them correctly or bring [24] them to such a form that the relations (11) are also fulfilled will already not be so easy. The problem, however, is analogous to the previous one if we consider the equation after substitution of the n-manifold expansion as the univariate polynomial with respect to some V_1,

$$E_1 = A_1(V_1, \ldots, V_n; x, S, S_x, \ldots)\, G_1(V_1, \ldots, V_n; x, S, S_x, \ldots) = 0$$

with the parameters V_i ($i \neq 1$), x, S, S_x, ... All the aforementioned is valid in this case. Although the theorem on a resultant is not efficient here from the computational viewpoint (in particular, it does not allow one to throw away trivial solutions like $V_i = V_j$) and leads to a huge algebra in contrast to the Gröbner basis method, it postulates that the resulting condition for the parameters exists and is presented by one expression, say

$$E_2(V_2, \ldots, V_n; x, S, S_x, \ldots) = 0.$$

In turn this equation can be treated as the polynomial in V_2 and

$$E_2 = A_2(V_2, \ldots, V_n; x, S, S_x, \ldots)\, G_2(V_2, \ldots, V_n; x, S, S_x, \ldots) = 0$$

again and so on. This process can be continued up to $m + 1$,

$$E_{m+1}(V_{m+1}, \ldots, V_n; x, S, S_x, \ldots) = 0. \tag{13}$$

If $m < n$ there will be the m constraints to the n functions $\{V_1, \ldots, V_n\}$, the functions $\{V_{m+1}, \ldots, V_n\}$ remain 'free' with the arbitrary parameters due to (2), and we should further equate to zero the related coefficients at their products in (13). The case $m = n$

$$E_{n+1}(x, S_1, \ldots, S_n, \ldots) = 0$$

returns us to the problem studied previously. The relations (9) constructed in this manner are verified as before, and the system (11) takes the triangular form in so doing.

3. A singular manifold with a constraint. Some examples

Below the application for one-manifold cases of the theory and the approach developed in the previous section is demonstrated for some reductions of the KdV and MKdV equations. The use of a resultant is reasonable and effective in these cases.

As is well known, the KdV and MKdV equations are closely allied to one another within the framework of the singular manifold method. To be precise, the singular manifold equation

$$C - S - \lambda = 0 \qquad \lambda = \text{constant}$$

and the truncated expansion

$$u(x,t) = -2V^2 + \tfrac{1}{6}(C - 4S) = V_x - V^2 + \tfrac{1}{6}\lambda$$

are related to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

And in so doing, equations (3) and (4) take the form of the MKdV equation and the KdV one again, namely

$$V_t + \lambda V_x - 6V^2 V_x + V_{xxx} = 0$$

$$S_t + \lambda S_x + 3SS_x + S_{xxx} = 0.$$

Naturally, one could expect that the use of the singular manifold method to their reductions could give rise to analogous relations and maps. Paradoxical as it may seem, it only leads to particular solutions. To overcome this in [13] the authors proposed simply to pick up suitable terms, and in [14] an ansatz was applied with the same purpose. However, the problem can be solved easily and straightforwardly.

Example 1. Consider the following travelling wave reduction ($z = x - ct$, $V_z = V_x = -V^2 - S/2$):

$$u_z^2 - u^4 - cu^2 + bu + a = 0 \qquad a, b, c = \text{constant} \tag{14}$$

of the MKdV twice integrated, one time after multiplication by u_z. Substituting the truncated 'expansion'

$$u = \pm V \tag{15}$$

into (14), we have the equation

$$E = 4(S - c)V^2 \pm 4bV + (4a + S^2) = 0.$$

The appropriate resultant is of the form

$$\operatorname{Res}\left(E, \frac{\mathrm{d}}{\mathrm{d}z}E\right) = \tfrac{1}{16}(S - c)\big[S^4 - 4cS^3 + 4(c^2 + 2a)S^2 + 8(b^2 - 2ac)S + 16a^2\big]\big[S_z^2 + S^3 - cS^2 + 4aS - (4ac + b^2)\big].$$

Ignoring the first trivial factors, one has

$$S_z^2 + S^3 - cS^2 + 4aS - (4ac + b^2) = 0. \tag{16}$$

Provided that S satisfies this relation, E and $\mathrm{d}E/\mathrm{d}z$ have the following common divisor:

$$G = 2(S - c)V - S_z - b.$$

(E can be rewritten in the equivalent form

$$E = \left(\frac{2S_z + G}{S - c}\right) G$$

in view of (16).) In so doing, the condition (9) is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z} G = -\frac{(2b + G)}{2(S - c)} G.$$

Finally, finding the connection between V and S from the relation $G = 0$, one arrives at the following map:

$$u = \pm \frac{S_z + b}{2(S - c)}$$

between (14) and the analogous reduction of the KdV (16) as might be expected.

Example 2. Next consider the analogous travelling wave ($z = x - ct$) KdV reduction after one time integration,

$$u_{zz} + 6u^2 - cu + b = 0 \qquad b, c = \text{constant}. \tag{17}$$

With the substitution

$$u = -V^2 + \tfrac{1}{12}(c - 4S) \tag{18}$$

one obtains, respectively,

$$E = 24S_z V - 8S_{zz} + 4S^2 + 24b - c^2 \tag{19}$$

$$\begin{aligned} \operatorname{Res}\left(E, \frac{\mathrm{d}}{\mathrm{d}z}E\right) = {} & \tfrac{1}{576} S_z\big[-192 S_{zzz} S_z + 128 S_{zz}^2 + 8(c^2 - 24b - 4S^2) S_{zz} \\ & -96 S_z^2 S - 16 S^4 + 8(c^2 - 24b^2) S^2 - 576 b^2 + 48 bc^2 - c^4\big] = 0. \end{aligned} \tag{20}$$

It is clear that (19) and the common divisor G sought are identical, $G = E$. In this case, since the equation to S is of third order, we should investigate its intermediate integrals. In the general case finding all such integrals for a differential equation is a complicated enough problem. However, it can be fully solved algorithmically for ODEs and the above-mentioned integrals of the polynomial type [25]. (In reality, a polynomial with respect to only the highest derivatives is enough frequently.) Here, for our purposes we can restrict ourselves to the even more special type

$$S_{zz} = F(S_z, S, z). \tag{21}$$

In view of the condition for S from (20) the determining equation for F is of the form

$$\begin{aligned} & 192\big(S_z F F_{S_z} + S_z^2 F_S + S_z F_z\big) - 128 F^2 + 8(4S^2 + 24b - c^2) F \\ & \qquad +96 S S_z^2 + 16 S^4 + 8(24b - c^2) S^2 + 576 b^2 - 48 bc^2 + c^4 = 0 \end{aligned}$$

whence, taking into account the possible dominant terms with $S_z F F_{S_z}$, $S_z^2 F_S$, F^2 and $S S_z^2$, it immediately follows that for F polynomial in S_z,

$$F = f_2(S, z) S_z^2 + f_1(S, z) S_z + f_0(S, z).$$

After that, equating to zero the coefficients at the powers of S_z, one arrives at a system for f_0, f_1 and f_2,

$$\begin{aligned}
&3f_{0S} + 4f_0^2 = 0\\
&3f_{1S} + 5f_0 f_1 + 3f_{0_x} = 0\\
&c^2 f_0 - 8f_1^2 - 12S - 24bf_0 - 24f_{2S} - 24f_{1_z} - 4(4f_2 + S^2)f_0 = 0\\
&(24b - c^2)f_1 + 24f_{2_z} - 4(2f_2 - S^2)f_1 = 0\\
&(24b - c^2 + 16f_2 + 4S^2)(24b - c^2 - 8f_2 + 4S^2) = 0.
\end{aligned}$$

These equations can be further simplified [26, 27] and reduce to the following:

$$\begin{aligned}
&f_0 = \tfrac{1}{16}(c^2 - 24b - 4S^2)\\
&f_1 = 0\\
&f_{2S} = -\tfrac{4}{3}f_2^2\\
&f_{2x} = 0
\end{aligned}$$

so that (21) takes the form

$$S_{zz} - \frac{3S_z^2}{4(S+\varphi)} + \frac{S^2}{4} + \frac{24b - c^2}{16} = 0 \qquad \varphi = \text{constant}. \tag{22}$$

In particular, at $\varphi \to \infty$ in (22) one has the equation analogous to (17)

$$S_{zz} + \tfrac{1}{4}S^2 4 + \tfrac{3}{2}b - c^2 = 0 \tag{23}$$

and the map between equations (17) and (23)

$$u = -\left(\frac{c^2 - 24b - 4S^2}{16S_z}\right)^2 + \frac{(c - 4S)}{12}$$

as a consequence of taking into account (18), (19) and (23).

Example 3. As the final example, chose the popular P_{II} equation

$$v_{zz} - 2v^3 - zv + a = 0 \qquad a = \text{constant} \tag{24}$$

corresponding to one self-similar reduction

$$\begin{aligned}
u &= \frac{1}{(3t)^{1/3}} v(z)\\
z &= \frac{x}{(3t)^{1/3}}
\end{aligned} \tag{25}$$

(one time integrated) of the MKdV [28, 29]

$$u_t - 6u^2 u_x + u_{xxx} = 0.$$

The equation $V_x = -V^2 - S/2$ conserves its type so that $V_z = -V^2 - S/2$ for

$$\begin{aligned}
V &\to \frac{1}{(3t)^{1/3}} V(z)\\
S &\to \frac{1}{(3t)^{2/3}} S(z)
\end{aligned} \tag{26}$$

and in view of (15)

$$E = \pm 2(S - z)V \mp S_z + 2a \tag{27}$$

so that the resultant is as follows:

$$\mathrm{Res}\left(E, \frac{\mathrm{d}}{\mathrm{d}z}E\right) = \tfrac{1}{4}(S - z)\left[2(S - z)S_{zz} - S_z^2 + 2S_z + 2S^3 - 4zS^2 + 2z^2 S + 4a(a \mp 1)\right].$$

It is easy to verify that the related equation to S

$$2(S - z)S_{zz} - S_z^2 + 2S_z + 2S^3 - 4zS^2 + 2z^2 S + 4a(a \mp 1) = 0 \tag{28}$$

is the first integral of the associated self-similar reduction (equations (25) and (26))

$$S_{zzz} + 3SS_z - zS - 2S = 0$$

of the KdV. In this case the common divisor G for E and $\mathrm{d}E/\mathrm{d}z$ is E (27) itself, so that (9) is satisfied automatically. As a result, we have the map between two associated reductions of the MKdV (24) and KdV (28) [30],

$$u = \frac{-2a \pm S_z}{2(S - z)}. \tag{29}$$

This relation can be used to derived the mapping of (24) into itself as well. Really, the KdV and MKdV equations are linked via the Miura transformation [31]

$$S = 2(u'_x - u'^2) \tag{30}$$

such that

$$S_t + 3SS_x + S_{xxx} = 2\left(-2u' + \frac{\partial}{\partial x}\right)\left(u'_t - 6u'^2 u'_x + u'_{xxx}\right)$$

i.e. the MKdV equation is the intermediate integral for the equation obtained after the substitution. The transformation (30) again conserves the type for the above similarity, and proceeding for the intermediate integral after the substitution of (30) into (28) in the same manner as in the last example, one obtains

$$v'_{zz} - 2v'^3 - zv' + a' = 0 \qquad a' \mp a + 1 = 0. \tag{31}$$

(One could use the form of the differential operator associated with the Miura transformation to simplify the calculations.) And from (29) and (30) one has finally

$$v = -v' \mp \frac{a \pm a'}{2v'_z - 2v'^2 - z}.$$

This auto-transformation from (31) to (24) was derived for the P_{II} equation in the work [32].

4. Expansions with several constrained manifolds. Some examples

In this section some examples with two-manifold truncated series are demonstrated. In so doing, such manifolds may correspond to various singular branches. In contrast to the use of a resultant, the Gröbner basis technique is highly effective for the investigation of multivariate polynomials in this case and gives rise to the expressions sought within one or two steps.

Example 4. We will consider the ODE

$$cu_x - u_{xx} + 2u^3 + bu + a = 0 \qquad a, b, c = \text{constant} \tag{32}$$

corresponding ($x = y + ct$) to the following equation [33]:

$$u_t - u_{yy} + 2u^3 + bu + a = 0 \tag{33}$$

and show that some interesting results can be obtained for it.

Equation (32) has two opposite singular branches. Let us consider the following truncated expansion:

$$u = V_1 - V_2 + w(x). \tag{34}$$

In other words, one assumes that the infinite series could be presented in that manner. Therewith the first manifold function V_1 is associated with the principal part as usual, while the second one corresponds to the rest of the Laurent series, has no singularity on the same manifold, and could in turn be expanded into terms of V_1,

$$V_2 = w_0(x) + w_1(x)V_1^{-1} + \cdots \tag{35}$$

or vice versa and

$$V_1 = W_0(x) + W_1(x)V_2^{-1} + \cdots \tag{36}$$

respectively.

Inserting (34) into (32), one has

$$\begin{aligned} E = {} & -6V_1^2V_2 + (6w - c)V_1^2 + 6V_1V_2^2 - 12wV_1V_2 + (b - S^- - S^+ + 6w^2)V_1 \\ & +(6w + c)V_2^2 + (S^+ - S^- - b - 6w^2)V_2 \\ & +S_x^- - w_{xx} + cw_x + a + bw - cS^- + 2w^3 = 0 \end{aligned} \tag{37}$$

and as a consequence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}E = {} & 12V_1^3V_2 + 2(c - 6w)V_1^3 + 12wV_1^2V_2 + (6w_x - b - 2S^- + 4S^+ - 6w^2)V_1^2 \\ & -12V_1V_2^3 + 12wV_1V_2^2 + 12(S^- - w_x)V_1V_2 \\ & +(-S_x^- - S_x^+ + 12ww_x + cS^- + cS^+ - 12S^-w)V_1 \\ & -2(c + 6w)V_2^3 + (6w_x + b - 2S^- - 4S^+ + 6w^2)V_2^2 \\ & +(S_x^+ - S_x^- - 12ww_x + cS^- - cS^+ + 12S^-w)V_2 + S_{xx}^- - cS_x^- \\ & -w_{xxx} + cw_{xx} + bw_x + 6w^2w_x - bS^- + 2S^-S^+ - 6S^-w^2 = 0 \end{aligned} \tag{38}$$

where

$$S^- = S_1 - S_2 \qquad S^+ = S_1 + S_2. \tag{39}$$

Although constraints like $H(V_{1,2}; x, S_1, S_2, \ldots) = 0$ are not allowed here, we cannot yet equate coefficients at the various powers V_1, V_2 and their products to zero to obtain the relations for S_1 and S_2, because one admits some linkage between V_1 and V_2. Or, in other words, V_2 depends explicitly on V_1 and can be expressed in terms of it and vice versa. From this point of view, both equations (37) and (38) are of the type

$$\left(\sum_{i=1}^{+\infty} a_i^{pr_2} V_{r_2}^{-i}\right) V_{r_1}^p + \left(\sum_{i=0}^{+\infty} a_i^{p-1,r_2} V_{r_2}^{-i}\right) V_{r_1}^{p-1} + \cdots + \sum_{i=0}^{+\infty} a_i^{0r_2} V_{r_2}^{-i} = 0$$

$$p = 2, 3 \qquad (r_1, r_2) = (1, 2), (2, 1)$$

$$a_i^{pr_{1,2}} = a_i^{pr_{1,2}}\,(x, S_1, S_2, \ldots)$$

such that a_i^{pr} are unknown as yet, and the above-mentioned procedure is impossible. We can, however, reduce our system via the Gröbner basis technique to another form which is equivalent but such that at least one equation will be a univariate polynomial with respect to V_1 or V_2, or it will at least be of the form

$$b\,(x, S_1, S_2, \ldots)\,V_r^p + \cdots = 0 \qquad p \in N \quad r \in \{1, 2\}$$

so that $b\,(x, S_1, S_2, \ldots)$ could be rigorously set to zero.

Return to the system (37), (38) and construct its first reduction. Taking into account (37) with the highest product $V_1^2 V_2$ to eliminate the terms with $V_1^3 V_2$ and $V_1^2 V_2$ from (38), one has the following equation:

$$\begin{aligned}&(18w_x + 3b + c^2 - 6cw - 12S^- + 6S^+ + 18w^2)V_1^2\\ &\quad +12(cw - 3w_x + 2S^-)V_1 V_2 + (3S_x^- - 3S_x^+ - 6w_{xx} + 6cw_x + 36ww_x\\ &\quad +6a - bc + 6bw - 2cS^- + 4cS^+ - 6cw^2 - 36S^- w + 12w^3)V_1\\ &\quad +(18w_x - 3b - c^2 - 6cw - 12S^- - 6S^+ - 18w^2)V_2^2\\ &\quad +(3S_x^- + 3S_x^+ - 6w_{xx} + 6cw_x - 36ww_x + 6a + bc + 6bw - 2cS^- - 4cS^+\\ &\quad +6cw^2 + 36S^- w + 12w^3)V_2 + 3S_{xx}^- - 4cS_x^- - 3w_{xxx} + 4cw_{xx}\\ &\quad +3bw_x - c^2 w_x + 18w^2 w_x - ac - bcw - 3bS^- + c^2 S^- - 2cw^3\\ &\quad +6S^- S^+ - 18S^- w^2 = 0.\end{aligned} \tag{40}$$

Hence our system will be equivalent to the set of two equations, (38) and (40). (In fact, we have calculated the special type S-polynomial using the LEX order. In so doing, one of the initial equations (37) can be finally removed as a so-called superfluous polynomial.) The process could, of course, be continued to derive the triangular system, however, this is already enough.

Let us look to (40) attentively. In view of the aforementioned, in particular the requirements (35) and (36), the coefficients at V_1^2 and V_2^2 must be equal to zero or

$$S^- = \tfrac{1}{2}(3w_x - cw) \qquad S^+ = -\tfrac{1}{6}(3b + c^2 - 18w^2). \tag{41}$$

After that (40) takes the form

$$\begin{aligned}&(-9w_{xx} + 9cw_x + 36a - 18bc + 36bw - 4c^3 + 6c^2 w + 72w^3)V_1\\ &\quad +(-9w_{xx} + 9cw_x + 36a + 18bc + 36bw + 4c^3 + 6c^2 w + 72w^3)V_2\\ &\quad +3(3w_{xxx} - 7cw_{xx} - 12bw_x + 2c^2 w_x - 72w^2 w_x\\ &\quad -2ac + 4bcw + 32cw^3) = 0\end{aligned}$$

so that further simplifications are possible, and we should already equate to zero the coefficients at V_1, V_2 and then the rest as well. As a result, one has the following equation ($v = w/2$):

$$cv_x - v_{xx} + 2v^3 + \left(4b + \tfrac{2}{3}c^2\right)v + \left(8a - 4bc - \tfrac{8}{9}c^3\right) = 0 \tag{42}$$

provided that

$$c = 0 \qquad \text{or} \qquad a = 9b + 2c^2 = 0. \tag{43}$$

Another equation (37) from the system defines the constraint to V_1 and V_2,

$$\begin{aligned}G = &-216V_1^2 V_2 + 36(3v - c)V_1^2 + 216V_1 V_2^2 - 216vV_1 V_2\\ &\quad +3(3cv + 27v^2 - 9v_x + 18b + 2c^2)V_1 + 36(c + 3v)V_2^2\\ &\quad +3(3cv - 27v^2 - 9v_x - 18b - 2c^2)V_2\\ &\quad -9cv_x + 108a - 36bc + 54bv - 8c^3 + 15c^2 v + 27v^3 = 0\end{aligned} \tag{44}$$

such that E and $\mathrm{d}E/\mathrm{d}x$ can be rewritten as

$$E = \tfrac{1}{9}G$$

$$\frac{\mathrm{d}}{\mathrm{d}x}E = \tfrac{1}{27}(c - 6V_1 - 6V_2)G.$$

In summary we have obtained the one-parametric mapping

$$u = V_1 - V_2 + \tfrac{1}{2}v(x) \tag{45}$$

between equations (32) and (42) of the same form. In so doing, taking into account relations (39) and (41), the functions V_1 and V_2 are determined by the equations

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{1,2} = -V_{1,2}^2 - \tfrac{1}{24}\big(\pm 9v_x - 6b - 2c^2 \mp 3cv - 9v^2\big) \tag{46}$$

with the additional constraints (44) on them and (43) on the parameters.

The relation (45) between u and v represents the mapping of (32) into itself (for the latest choice of the parameters in (43) the equations for them coincide, and in the first case equation (42) is reduced exactly to the form (32) by the stretching $v \to 2v$, $x \to \pm x/2$) and can be applied for multiplication of already known solutions.

For instance, starting with the trivial solution $v_0 =$ constant such that $2v_0^3 + 4bv_0 + 8a = 0$ for the case $c = 0$, one has from (46)

$$V_1 = \frac{k}{2}\left(\frac{\mathrm{e}^{kx+\varphi} - 1}{\mathrm{e}^{kx+\varphi} + 1}\right)$$

$$V_2 = \frac{k}{2}\left[\frac{(v_0 + k)\mathrm{e}^{kx+\varphi} - (v_0 - k)}{(v_0 + k)\mathrm{e}^{kx+\varphi} + (v_0 - k)}\right]$$

$$\varphi = \text{constant} \qquad k^2 = b + \tfrac{3}{2}v_0^2 \neq 0$$

or

$$V_1 = \frac{1}{x + \varphi}$$

$$V_2 = \frac{1}{x + \varphi + 2/v_0}$$

$$\varphi = \text{constant} \qquad b + \tfrac{3}{2}v_0^2 = 0$$

with regard to the constraint (44) for them. The resulting expression for u is as follows:

$$u = \frac{-2k\mathrm{e}^{kx+\varphi}}{\big(\mathrm{e}^{kx+\varphi} + 1\big)\big[(v_0 + k)\mathrm{e}^{kx+\varphi} + (v_0 - k)\big]} + \frac{v_0}{2}$$

or, respectively,

$$u = \frac{2/v_0}{(x + \varphi)\,(x + \varphi + 2/v_0)} + \frac{v_0}{2}.$$

In a similar manner, starting from $v_0 = 0$ for $a = 0$, $b = -2c^2/9$, one obtains the expressions

$$V_1 = \tfrac{1}{6}c\tanh\big(\tfrac{1}{6}cx + \varphi\big)$$

$$V_2 = -\tfrac{1}{6}c$$

$$\varphi = \text{constant} \qquad c \neq 0$$

and, as a result, the simple solution

$$u = \frac{c}{3\left(1 + \mathrm{e}^{-cx/3-\varphi}\right)}.$$

Further application of (45) to the solution sought leads to their phase shift. By this means in these concrete cases the mapping amounts to an addition of some fixed solution to an initial one. While it itself is not limited by some subclasses of the general solutions to (32) so that u from (45) would satisfy a fixed (not dependent on another, arbitrary, solution v) ODE of a lesser (here first) order than (32). In view of (42), (44)–(46) one has the relations for its first derivative

$$16u_x^2 - 32au - 16bu^2 - 16u^4 - v_x^2 + 16av + 4bv^2 + v^4 = 0 \tag{47}$$

$$16\left(9u_x^2 - 6cuu_x + c^2u^2 - 9u^4\right) - 9v_x^2 + 6cvv_x - c^2v^2 + 9v^4 = 0 \tag{48}$$

respectively, for both cases. (We can, for example, express V_1 or V_2 in terms of another singular function from (45) and then find the resultant for (44) and its derivative or vice versa differentiate (45) to introduce u_x and solve these two relations to eliminate V_1 and V_2 from (44).)

The first of them, equation (47), can be represented as

$$16u_x^2 - 32au - 16bu^2 - 16u^4 = v_x^2 - 16av - 4bv^2 - v^4 = \lambda$$
$$\lambda = \text{constant}$$

and the action of the mapping affects the phase parameter. The non-trivial cases, when the mappings from the degenerate solutions with the phase $\varphi = \infty$ take place and we obtain the new solutions, have been presented above. The second case (48) is more complicated because

$$16\left(9u_x^2 - 6cuu_x + c^2u^2 - 9u^4\right) = 9v_x^2 - 6cvv_x + c^2v^2 - 9v^4 = \lambda\mathrm{e}^{4cx/3}$$
$$\lambda = \text{constant} \qquad c \neq 0$$

here and is not reduced to a trivial transformation. (The case corresponding to the phase transformation ($\lambda = 0$) and including the action to the degenerate solution has been demonstrated above.)

Before proceeding further, it should be mentioned that in the example just considered the resultant of E and $\mathrm{d}E/\mathrm{d}x$ presents the polynomial of the sixth order in V_1 (or V_2), involving from 4183 terms! And to give rise to the same results it would be necessary to solve the related overdetermined set of the huge differential equations for S_1, S_2 and w. That is very hard even applying the special theory and computer methods for such systems [26, 27]. While above, effectively using the nature of the functions V_1 and V_2, for the same purposes, we have practically dealt with only several simple enough algebraic relations.

However, in so doing, the constraint sought has been represented by the second-order polynomial, so that it has already been obtained in an early stage. In other cases it may be necessary to investigate polynomials of lower orders, and further calculations may appear to be noticeably complicated. While it is obvious that, on the one hand, possible orders of similar polynomials are fixed by the order of E, and, on the other hand, only some of their configurations with respect to V_i are compatible with the differential equations (2) for singular functions. As a result, if needed we could simply determine such possible configurations for an equation of interest first and then start our investigation by substituting any of them for E and directly equating coefficients for the products of V_i to zero. Although the first step nevertheless demands the use of the Gröbner basis technique, it is limited by the only reduction for each configuration. In this manner, in some cases calculations can be split beforehand into several branches, and superfluous ones could be avoided as a result. In appendix A such configurations for (37) are presented together with necessary details as an example. In that case, however, our direct strategy 'from top to bottom' is the most effective and economic, but for other equations the reverse may appear to be more reasonable.

Example 5. The obvious question arises of whether the previous use of the opposite branches is always essential for success? To answer this question consider briefly another example with the following ODE:

$$uu_{xx} - \alpha u_x^2 + a_0 + a_1 uu_x + a_2 u^2 + a_3 u + a_4 = 0 \qquad \alpha, a_i = \text{constant}. \tag{49}$$

A number of dissipative systems are reduced to the equations of this family [34, 35]. For $\alpha = \{2, \frac{3}{2}, \frac{4}{3}, \ldots\}$ and any a_i it obviously possesses the Painlevé property. But to derive something concrete for (49) is not so obvious. Its full investigation within the approach with several manifolds is highly complicated as well because of its nonlinearity, and because already first steps lead to overdetermined systems of differential equations to unknown functions in the related singular expansions and S_j. However, even considering the simplest possible configurations (A1) for the equation $E = 0$ obtained from (49) after the substitution of the simplest expansion ($\alpha = 2$)

$$u = w_1 V_1 + w_2 V_2 + w(x) \qquad w_1, w_2 = \text{constant} \tag{50}$$

and equating the coefficients in the powers $V_{1,2}^3$, $V_{1,2}^2$, $V_{1,2}$ and the free coefficient there to zero, one has

$$a = b = 0$$

$$c = \tfrac{1}{4}a_2$$

$$w = 0$$

$$S_1 = S_2 = -\tfrac{1}{2}a_2$$

with the additional relations for w_1, w_2 and a_i

$$(w_1 + w_2)^2 a_2^2 + 4a_4 = 0$$

$$3a_2(w_1 + w_2) - 2a_0 = 0$$

$$a_1 = a_3 = 0.$$

So that (50) corresponds to the following general solution:

$$u = \frac{a_0}{3\sqrt{a_2}} \tanh\left(\tfrac{1}{2}\sqrt{a_2}x + c_1\right) + c_2 \cosh\left(\sqrt{a_2}x + 2c_1\right) \qquad c_{1,2} = \text{constant}$$

to (49) at $a_{1,3} = a_0^2 + 9a_4 = 0$.

By this means the consideration of the two arbitrary branches in the singular expansion has been essential here.

There are a number of problems which are difficult for the traditional singular manifold method. Among them are systems with perturbations and uncoupling of sets of differential equations. Really, in the former case the postulate 'near singularities', as such, renders accurate consideration of a small parameter doubtful, and in the latter the use of only one singular function beforehand assumes a linkage of the functions sought.

Example 6. Consider now the following system of ODEs with a small parameter and perturbations of the most general type quadratic with respect to each of the functions

$$u_{1x} = -u_1^2 - \tfrac{1}{2}a_0 + \varepsilon\left[\left(a_1 + a_2u_1 + a_3u_1^2\right)u_2^2 + \left(a_4 + a_5u_1 + a_6u_1^2\right)u_2\right] \quad (51)$$

$$u_{2x} = -u_2^2 - \tfrac{1}{2}b_0 + \varepsilon\left[\left(b_1 + b_2u_2 + b_3u_2^2\right)u_1^2 + \left(b_4 + b_5u_2 + b_6u_2^2\right)u_1\right] \quad (52)$$

$$|\varepsilon| \ll 1 \qquad a_i, b_i = \text{constant} \qquad i = \overline{0, 6}.$$

Other than their important role in chemical kinetics, when the variables correspond to various species, and other fields of physics (e.g. turbulence and plasma physics), similar types of systems arise in the study of dynamics of ensembles from weakly interacting solitonic waves (so-called soliton gases and soliton grids) in integrable and, especially, non-integrable models. At $\varepsilon = 0$, we have description (x-part) for two such non-interacting solutions typical within the framework of the singular manifold technique, and perturbations appear from nonlinear terms if one admits wave overlapping.

It is known that such interacting ensembles assume various dynamics, from regular behaviour up to formations of bound states and chaotic motion even with two or three components [36–39].

Although to derive a set of equations for corrections within the framework of the direct perturbation theory is very simple, to solve such equations or even to reveal basic features of the solutions is very difficult even in the simplest cases. While it is known that some such systems or single equations, both ordinary and partial differential ones, can be transformed to an unperturbed or simpler form [40–42]. Let us investigate (51) and (52) from this viewpoint confining ourselves to the first order of the perturbation theory and use the related two-manifold expansions

$$u_1 = V_1 + \varepsilon\left(w_{11}V_1 + w_{12}V_2 + w_{10}\right) + \mathrm{o}(\varepsilon) \quad (53)$$

$$u_2 = V_2 + \varepsilon\left(w_{21}V_1 + w_{22}V_2 + w_{20}\right) + \mathrm{o}(\varepsilon) \quad (54)$$

$$w_{ij} = w_{ij}(x) \qquad i = \{1, 2\} \quad j = \overline{0, 2}.$$

(The zero-order part is obvious, and the first order one is dictated by the balance between the dominant terms from the unperturbed part and perturbation.) The list (A1)–(A3) exhausts all permutable configurations for the resulting equations:

$$\begin{aligned}&2a_3V_1^2V_2^2 + 2a_6V_1^2V_2 - 2w_{11}V_1^2 + 2a_2V_1V_2^2 + 2(a_5 - 2w_{12})V_1V_2\\&\qquad -2V_1(w_{11x} + 2w_{10}) + 2(a_1 + w_{12})V_2^2 + 2V_2(a_4 - w_{12x})\\&\qquad -2w_{10x} + a_0w_{11} + b_0w_{12} + S_{11} = 0\end{aligned} \quad (55)$$

$$\begin{aligned}&2b_3V_1^2V_2^2 + 2b_2V_1^2V_2 + 2(b_1 + w_{21})V_1^2 + 2b_6V_1V_2^2\\&\qquad +2(b_5 - 2w_{21})V_1V_2 + 2V_1(b_4 - w_{21x}) - 2w_{22}V_2^2 - 2V_2(w_{22x} + 2w_{20})\\&\qquad -2w_{20x} + a_0w_{21} + b_0w_{22} + S_{21} = 0.\end{aligned} \quad (56)$$

Starting with the simplest of them (A1), one has further from (55) and (56)

$$w_{11} = (a_3a + a_6)a$$

$$w_{12} = -(a_1 + a_2b + a_3b^2)$$

$$\begin{aligned}w_{10} = \tfrac{1}{4}\big[&4a_3a^3 + 2(a_2 + a_6)a^2 + 4a_3ab^2 + 4a_2ab + 8a_3ac\\&+2(2a_1 + a_3b_0 + a_5)a + 4a_6c + a_6b_0\big]\end{aligned}$$

$$w_{21} = -(b_3a^2 + b_2a + b_1)$$

$$w_{22} = (b_3b + b_6)b$$

$$w_{20} = \tfrac{1}{4}\big[4b_3a^2b + 4b_2ab + 4b_3b^3 + 2(b_2 + b_6)b^2 + 8b_3bc$$
$$+2(a_0b_3 + 2b_1 + b_5)b + 4b_6c + a_0b_6\big]$$

and for S_1 and S_2

$$S_1 = a_0 + \tfrac{1}{2}\varepsilon\big[-12a_3a^4 - 4(a_2 + a_6)a^3 - 36a_3a^2c$$
$$-2(a_0a_3 + 2a_1 + 4a_3b_0 + a_5)a^2 + 4(a_2 + a_6)ab^2 - 24a_3abc$$
$$+4(2a_1 + a_5 - a_0a_3)ab - 12(a_2 + a_6)ac$$
$$-2(a_0a_2 + a_0a_6 + a_2b_0 + a_6b_0)a - 12a_3b^2c - 12(a_2 + a_6)bc$$
$$-12a_3c^2 - 6(2a_1 + a_3b_0 + a_5)c - b_0(a_3b_0 + a_5)\big]$$

$$S_2 = b_0 + \tfrac{1}{2}\varepsilon\big[4(b_2 + b_6)a^2b - 12b_3a^2c - 24b_3abc$$
$$+4(2b_1 + b_5 - b_0b_3)ab - 12(b_2 + b_6)ac - 12b_3b^4 - 4(b_2 + b_6)b^3$$
$$-36b_3b^2c - 2(4a_0b_3 + b_0b_3 + 2b_1 + b_5)b^2 - 12(b_2 + b_6)bc$$
$$-2(a_0b_2 + a_0b_6 + b_0b_2 + b_0b_6)b - 12b_3c^2 - 6(a_0b_3 + 2b_1 + b_5)c$$
$$-a_0(a_0b_3 + b_5)\big]$$

together with the following restrictions for the functions a and b in (A1):

$$2(b_2 + b_6)a^2 + 2(2b_1 + b_5 - b_0b_3)a + 2b_4 - b_0b_3 = 0 \tag{57}$$

$$2(a_2 + a_6)b^2 + 2(2a_1 + a_5 - a_0a_3)b + 2a_4 - a_0a_3 = 0. \tag{58}$$

If a or b are fixed constants then (53) and (54) in view of (A4)–(A6) would describe just a particular solution. So that we should set

$$a_6 = -a_2 \qquad a_5 = a_0a_3 - 2a_1 \qquad a_4 = \tfrac{1}{2}a_0a_2$$

$$b_6 = -b_2 \qquad b_5 = b_0b_3 - 2b_1 \qquad b_4 = \tfrac{1}{2}b_0b_2$$

here for the general solution. By this means the functions a, b and c are arbitrary as yet, and we will try to evolve the case when in our approximation (51) and (52) can be reduced by (53) and (54) to their unperturbed form and demand additionally that

$$S_1, S_2 = \text{constant}.$$

This problem is again reduced to the calculation of the set of differential consequences for these two equations (57) and (58), and finding the invariant constraints not restricting a, b, c to fixed constants. All our previous techniques are applied for this, and one has the following conditions after the tiresome but straightforward algebra

$$c = -a^2 - \tfrac{1}{2}b_0$$

$$b = -a$$

$$b_0 = a_0$$

and consequently

$$V_1V_2 + (V_2 - V_1)a + \tfrac{1}{2}a_0 = 0$$

and, as a result, for u_1 and u_2

$$u_1 = V_1 + \varepsilon\left[a_3\frac{\left(V_1V_2 + \frac{1}{2}a_0\right)}{V_1 - V_2} - a_2\left(V_1V_2 + \tfrac{1}{2}a_0\right) - a_1V_2 + \tfrac{1}{4}a_0a_2\right] + \mathrm{o}(\varepsilon)$$

$$u_2 = V_2 - \varepsilon\left[b_3\frac{\left(V_1V_2 + \frac{1}{2}a_0\right)}{V_1 - V_2} - b_2\left(V_1V_2 + \tfrac{1}{2}a_0\right) - b_1V_1 + \tfrac{1}{4}a_0b_2\right] + \mathrm{o}(\varepsilon)$$

provided that

$$S_1 = a_0 + \varepsilon a_0 \left(\tfrac{1}{2}a_0 a_3 + a_1\right) + \mathrm{o}(\varepsilon)$$
$$S_2 = a_0 + \varepsilon a_0 \left(\tfrac{1}{2}a_0 b_3 + b_1\right) + \mathrm{o}(\varepsilon)$$

or

$$V_1 = \sqrt{-\tfrac{1}{2}S_1}\tanh\left(\sqrt{-\tfrac{1}{2}S_1}x + \varphi_1\right) + \mathrm{o}(\varepsilon)$$
$$V_2 = \sqrt{-\tfrac{1}{2}S_2}\tanh\left(\sqrt{-\tfrac{1}{2}S_2}x + \varphi_2\right) + \mathrm{o}(\varepsilon)$$
$$\varphi_1, \varphi_2 = \text{constant}.$$

For arbitrary values of a_0 and b_0 other simplifications can be obtained. For instance, the trivial case with the independent manifolds $c = 0$, $a = V_2$, $b = V_1$ leads to the following expressions for S_1, S_2 and the solutions u_1, u_2:

$$S_1 = a_0 - \varepsilon\left[6a_3V_2^4 + 2a_3(a_0 + 2b_0)V_2^2 + \tfrac{1}{2}b_0(a_0a_3 + a_3b_0 - 2a_1)\right] + \mathrm{o}(\varepsilon)$$
$$S_2 = b_0 - \varepsilon\left[6b_3V_1^4 + 2b_3(b_0 + 2a_0)V_1^2 + \tfrac{1}{2}a_0(a_0b_3 + b_3b_0 - 2b_1)\right] + \mathrm{o}(\varepsilon)$$
$$u_1 = V_1 + \varepsilon\left[a_3V_1V_2^2 - a_2V_1V_2 + a_3V_2^3 + 2(a_0a_3 + a_3b_0 - 2a_1)V_2 - \tfrac{1}{4}a_2b_0\right] + \mathrm{o}(\varepsilon)$$
$$u_2 = V_2 + \varepsilon\left[b_3V_1^2V_2 - b_2V_1V_2 + b_3V_1^3 + 2(a_0b_3 + b_3b_0 - 2b_1)V_1 - \tfrac{1}{4}b_2a_0\right] + \mathrm{o}(\varepsilon).$$

In other words, the interactions can be reduced to the weak wavenumber modulation determined by another component only.

The analogous analysis for other configurations (A2) and (A3) also gives rise to positive results, however, they really add nothing new to the results just found.

The approach presented above can be applied with success to a number of other nonlinear ordinary and partial differential equations (one will need to consider $\partial E/\partial x$ and $\partial E/\partial t$ in the latter case). Some time-dependent solutions of the initial equation (33) could be obtained in this manner.

5. Some remarks and further perspectives

In the previous sections it has been demonstrated that some problems arising in modern applications of the singular manifold method can be solved rigorously if one treats this technique more attentively. A critical reader, however, may have some questions: series of what form could be used in light of the preceding? Could one introduce terms like $V_1^{n_1}V_2^{n_2}$ $(n_1, n_2 \in N)$ there? What can the maximal order of such series be with respect to V_i if the related coefficient depends on other functions V_j $(j \neq i)$? How many manifolds are needed to describe the general solution? There is no constructive answer within the framework of the singular analysis. Moreover, the technique itself could be considered from another point of view.

Assume that there is a set M of functions $\{\ldots, e_i(x, t; \ldots, \tau_k, \ldots), \ldots\}$ finite or infinite such that

$$\frac{\partial e_i}{\partial x} = \sum_l a_{il}(x, t)e_l \tag{59}$$
$$\frac{\partial e_i}{\partial t} = \sum_l b_{il}(x, t)e_l \tag{60}$$
$$e_ie_j = \sum_l c_{ijl}(x, t)e_l. \tag{61}$$

The variables τ_k correspond to the arbitrary parameters associated with the differential equations (59) and (60); and e_{il}, b_{il}, c_{ijl} satisfy the compatibility conditions for (59)–(61). Then a polynomial nonlinear differential operator $E(x, t, u, u_x, u_t, \ldots)$ maps the linear space

$$L\{\ldots, e_i, \ldots\} = \left\{ \sum_j w_j(x,t) e_j(x, t; \ldots, \tau_k, \ldots) \right\}$$

into itself. Let there exist finite subsets W, $W_E \subset M$ with the related linear subspaces L_W and L_{W_E} such that

$$E[L_W] \subseteq L_{W_E}.$$

If one inserts an element $u \in L_W$ into the equation $E = 0$ and separates the variables, one has an overdetermined but compatible system of equations to $w_j(x,t)$, then such sets can be used for finding solutions of this equation. The fact that for special types of matrices $\{a_{ij}\}$, $\{b_{ij}\}$ and $\{c_{ijk}\}$ an infinite linear system (59) and (60) can be a representation of a finite-dimensional dynamic system [43, 44]. (For instance, the Riccati equations (2) and (3) correspond to the set $\{1, V, V^2, \ldots\}$.) It is clear now that the above-mentioned constraints correspond to the cases when one or several vectors (functions) e_i appear to be linearly dependent on others. In terms of nonlinear systems they correspond to special integrals or invariant manifolds [45].

All of these are not new in effect. First, the analogous concept, namely linear subspaces or sets on linear subspaces invariant under a nonlinear differential operator [46, 47], has been applied with success and has developed quickly for ten years. (See [48] for details on finding the form of such sets.) Second, in all of our examples one has finally been led to such systems taking into account other functions except $V_{1,2}$ in those singular expansions.

In fact it is not so tedious to find a suitable system for a differential equation of interest. For example, for the travelling wave reduction

$$-cu_x + u_{xx} + u^2 - u = 0 \tag{62}$$

of Fisher's equation [49], the auto-transformation can be formally constructed (see appendix B)

$$u = -6(V_1 - V_2)^2 + u'$$

and (we have set $a_3 = b_3 = V_3(x)$ in (B1) and (B3), (B4))

$$\frac{\mathrm{d}}{\mathrm{d}x} V_{1,2} = -V_{1,2}^2 + 5V_1V_2 + \tfrac{1}{5}(c + 3V_3)V_{1,2} + V_{2,1}V_3$$
$$\frac{\mathrm{d}}{\mathrm{d}x} V_3 = \tfrac{4}{5}V_3^2 + \tfrac{1}{5}cV_3 + \tfrac{1}{20}(50u' - 6c^2 - 25).$$

However, in so doing, the main problem is whether these auxiliary equations themselves can be integrated? While in [50] it was shown that such systems could be highly important from another viewpoint for understanding of the behaviour of nonlinear waves in PDEs, because special superposition properties and solitonic solutions may be associated with them.

In conclusion, it should be pointed out once more that the availability of the parameters τ_k is essential for all the aforementioned, because this leads to a separation of the functions e_i and coefficients. Also, in principle, proceeding in the manner demonstrated in section 3, it is possible to obtain some formulae for any ansatz. However, if such a substitution is not justified theoretically, its usefulness depends only on the luck of a researcher and the particular case. Singular manifold equations themselves can be useless. And even for the well known integrable NPDEs additional investigations are frequently necessary. For example, for the

Burgers equation $\{u_t + uu_x - u_{xx} = 0; u = -2V + C\}$ one has the system for S and C from the singular manifold equation

$$C_t - 2C_{xx} + CC_x - S_x = 0 \tag{63}$$

and (4). The suitable equation

$$C_t + \lambda C + 2CC_x - C_{xx} = 0 \qquad \lambda = \text{constant}$$

is found here among the intermediate integrals of the system (4) and (63).

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Appendix A

In accordance with the maximum order of (37) with respect to each singular function V_1 and V_2, only the following configurations (substitutions) could be permissible for it:

$$-V_1V_2 + aV_1 + bV_2 - ab + c = 0 \tag{A1}$$

$$V_1^2V_2 + bV_1V_2^2 + cV_1^2 + dV_2^2 + fV_1V_2 + gV_1 + hV_2 + q = 0. \tag{A2}$$

Here a, b, c, d, f, g, h and q are some new functions unknown beforehand. (One more second-order configuration

$$-V_1^2V_2^2 + aV_1^2V_2 + bV_1V_2^2 + cV_1^2 + dV_2^2 + fV_1V_2 + gV_1 + hV_2 + q = 0 \tag{A3}$$

is impossible owing to the absence of the suitable leading term $V_1^2V_2^2$ in (37).) In (A1) the new functions a, b and c have been introduced so that for $c = 0$ one has the trivial case with the independent manifolds

$$(V_1 - b)(V_2 - a) = 0.$$

To derive the relations to the coefficients in (A1) and (A2), it is necessary to construct the differential consequence $G_x = 0$ and exclude the leading terms with V_1 and V_2 taking into account the constraint $G = 0$ itself. Then, equating the coefficients at all products $V_1^{n_1}V_2^{n_2}$ to zero, one will obtain the differential equations and, maybe, additional algebraic restrictions to the above functions. By this means they should satisfy the following relations:

$$a_x = -a^2 - c - \tfrac{1}{2}S_2 \tag{A4}$$

$$b_x = -b^2 - c - \tfrac{1}{2}S_1 \tag{A5}$$

$$c_x = -2c(a + b) \tag{A6}$$

and

$$b_x = -b^2c + bc + 2bf - 2d$$

$$c_x = -bc^2 + c^2 + cf - g + \tfrac{1}{2}S_2$$

$$d_x = -bcd + bh + \tfrac{1}{2}bS_1 + cd + df$$

$$f_x = -bcf + bg + bS_2 + cf + f^2 - g - 2h + S_1$$

$$g_x = -bcg + cg + cS_1 + fg + \tfrac{1}{2}fS_2 - 2q$$

$$h_x = -bch + bq + ch + dS_2 + fh + \tfrac{1}{2}fS_1 - q$$

$$q_x = -bcq + cq + fq + \tfrac{1}{2}gS_1 + hS_2$$
$$b(b+1) = d(b+1) = 0$$

for (A1) and (A2), respectively.

Analogously, for PDEs G_t together with G_x should be calculated, and the t-parts should be found as well. For example, for (A1) the t-part is of the form

$$\begin{aligned} a_t &= C_2a^2 - C_{2x}a + C_1c + \tfrac{1}{2}(C_{2xx} + C_2S_2) \\ b_t &= C_1b^2 - C_{1x}b + C_2c + \tfrac{1}{2}(C_{1xx} + C_1S_1) \\ c_t &= c\big(2C_2a + 2C_1b - C_{1x} - C_{2x}\big). \end{aligned} \tag{A7}$$

Obviously, equations (A4)–(A6) and (A7) are compatible when (4) is satisfied.

Appendix B

One will try to construct the analogue of a WTC truncated expansion for (62) using instead of the Riccati equations (2), the following system:

$$\begin{aligned} V_{1x} &= -V_1^2 + a_1(x)V_1V_2 + a_2(x)V_1 + a_3(x)V_2 \\ V_{2x} &= -V_2^2 + b_1(x)V_1V_2 + b_2(x)V_2 + b_3(x)V_1. \end{aligned} \tag{B1}$$

For the constant parameters this system was investigated in [51] with the goal of a complete classification of two-dimensional quadratic systems from the Painlevé analysis viewpoint. The related series with respect to V_1 and V_2 is of the form

$$u = w_1(x)V_1^2 + w_2(x)V_1V_2 + w_3(x)V_2^2 + w_4(x)V_1 + w_5(x)V_2 + w(x) \tag{B2}$$

according to the dominant behaviour of the functions u in (62) and V_1 and V_2 in (B1). After inserting (B2) into the equation under consideration, the leading orders (the coefficient for the products V_1^4, $V_1^3V_2$, $V_1^2V_2^2$, $V_1V_2^3$ and V_2^4) give the set of the algebraic relations

$$w_1(w_1+6) = 0$$
$$2a_1b_1w_1 - 10a_1w_1 + b_1^2w_2 - 3b_1w_2 + 2w_1w_2 + 2w_2 = 0$$
$$4a_1^2w_1 + 4a_1b_1w_2 - 2a_1w_1 - 3a_1w_2 + 4b_1^2w_3 - 3b_1w_2 - 2b_1w_3 + 2w_1w_3 + w_2^2 + 2w_2 = 0$$
$$a_1^2w_2 + 2a_1b_1w_3 - 3a_1w_2 - 10b_1w_3 + 2w_2w_3 + 2w_2 = 0$$
$$w_3(w_3+6) = 0.$$

These relations can be solved, for example, via the Gröbner bases approach, so that, in particular, the following non-trivial sets for w_1, w_2 and w_3, the coefficients of the dominant terms in (B2), are determined:

$$\begin{aligned} (w_1, w_2, w_3) = \{&(0; -6, -1; 0), (0; 0, -24, 3, -6, 2, 6, 8; -6), \\ &(-6; -84, 12, 24, 20, 15, -12, 6; -6)\}. \end{aligned}$$

One choice, namely $(w_1, w_2, w_3) = (-6, 12, -6)$, leads us to a positive result for any value of c from (62). More precisely, in this case the above system also gives

$$a_1 = b_1 = 5$$

while the algebraic again equations in the next order (the coefficients at V_1^3, $V_1^2V_2$, $V_1V_2^2$ and V_2^3) give

$$w_{4,5} = \pm\tfrac{12}{5}(b_3 - a_3)$$
$$a_2 = \tfrac{1}{5}(c - 2a_3 + 5b_3)$$
$$b_2 = \tfrac{1}{5}(c - 2b_3 + 5a_3).$$

After that the rest of the orders already determine the differential equations for a_3 and b_3

$$a_{3x} = \tfrac{1}{20}\left(8a_3^2 - 4a_3b_3 + 4ca_3 + 12b_3^2 - 6c^2 + 50w - 25\right) \tag{B3}$$
$$b_{3x} = \tfrac{1}{20}\left(8b_3^2 - 4a_3b_3 + 4cb_3 + 12a_3^2 - 6c^2 + 50w - 25\right) \tag{B4}$$

together with the equation to w, which is obviously identical to the original one for u. Note here that since the system (B3) and (B4) is symmetrical, we could set $a_3 = b_3$ there for simplicity.

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